Dark current and beam loading in blowout regime of laser-plasma acceleration: A test-bed for reduced simulation methods

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Modern laser-plasma accelerators work in the blowout ("bubble") regime.

Accelerating structure is no more a periodic quasi-1D nonlinear wake. It is electron density BUBBLE!

- Projectile in gas
- Laser pulse in plasma

**Why blowout regime?**

1. Accelerating gradient strongly exceeds the cold wave breaking limit

2. Beam loading limitations are greatly relaxed [M. Tzoufras et al., *Phys. Plasmas* 16, 056705 (2009)]
   \[ \Rightarrow \text{Efficiency is enhanced} \]

3. In any transverse cross-section of the bucket, accelerating gradient is uniform

4. The entire bucket is focusing; focusing gradient is linear with radius: \( \varepsilon_{N,\perp} \) is thus preserved
   \[ \Rightarrow \text{Electron beam quality is expected to be preserved} \]

5. Electron are self-injected \( \Rightarrow \text{Expected technical simplicity} \)
Plasma electrons in the blowout regime

Electron cavity ("bubble") travels over the positive ion background at a relativistic speed.

Bubble is essentially a quasi-static structure:
Majority of plasma electrons, expelled by the radiation pressure and attracted by the charge separation force, are passing.

Electrons forming the dense shell surrounding the bubble are trapping candidates.
They follow the bubble for a long time, and become relativistic.

Bubble is observable in the lab

Z. Li et al., *PRL* **113**, 085001 (2014)

Experimental correlation:
Electron self-injection & generation of monoenergetic beams

Formation of the optical bullet
– probe light trapped and focused by the bubble
Two major physical processes critical for the electron beam quality:

• **Beam loading:**
  Compensation of the plasma wakefield with the own wakefield of the accelerated electron beam, leading to the reduction of the accelerating gradient

• **Dark current:**
  Self-injection of unwanted electrons through the acceleration process, leading to the emittance dilution and formation of the massive polychromatic electron energy tail
There is a number of confusions caused by the abuse of fully kinetic codes:

1. Beam loading terminates injection and leads to the formation of the QM beam
   [C. G. R. Geddes et al., *Nature* 431, 538 (2004); S. P. D. Mangles et al., *Phys. Plasmas* 14, 056702 (2007); ... ]

2. Beam loading causes elongation of the bubble and enforces continuous injection (dark current)
   [V. Malka et al., *Phys. Plasmas* 12, 056702 (2005); F. S. Tsung et al., *Phys. Plasmas* 13, 056708 (2006); ... ]

Clearing up these confusions calls for *the reduced models* capable of “turning off” the electron beam own wakes, while preserving accurate dynamics of the laser pulse and quasistatic bulk electron response.
Correct theory resolves the confusion as to the role of beam loading in termination of injection.

To saturate injection, the charge/current density must be at least an order of magnitude higher than in a quasi-monoenergetic bunch seen in experiments/simulations.

The physical origin of bubble expansion/continuous injection remains unanswered!

**First step:**

**Accurate theory of beam loading for the fully cavitated wake**
Further step: Reduced model – quasistatic PIC code with test particles

Conceptual framework for the self-injection/beam loading/dark current problem can be built using an exceptionally simple, efficient, and physically correct numerical model.

Fully relativistic quasi-static cylindrical PIC code WAKE [1]

1. Ponderomotive guiding center
   (e.g. time averaged over the laser cycle)

2. Extended paraxial
   (correct group velocity dispersion in a broad frequency range, no side-scatter; accurate description of pulse depletion and self-compression due to the wake excitation)

3. Plasma macroparticles are quasistatic and are pushed by the time-averaged ponderomotive force of the laser.

+ built-in test electron tracking module (full 3D, dynamic, non-averaged) [2,3]

[2] V. Malka ... A. Solodov, Phys. Plasmas 8, 2605 (2001);
**How good is the test-particle toolbox?**

**WAKE vs. CALDER-Circ**

**CALDER-Circ** — quasi-cylindrical, fully explicit 3D PIC code with a poloidal mode decomposition of fields and currents; with 3rd order macroparticles, numerical Cherenkov-free solver

→ computational efficiency of a fully explicit electromagnetic 2D PIC code


**COMPARATIVE EFFICIENCY:**

**WAKE:** 20 hours on a single-core 2.13 GHz Intel processor, 1 GB RAM

**CALDER-Circ:** 21 500 CPU hours (86 h on 250 cores) on “Titane” cluster at CEA/DIF/DAM, Bruyères-le-Châtel, France

WAKE runs $> 10^3$ times faster than CALDER-Circ and correctly captures all relevant physics of plasma wake evolution and dynamics of electron self-injection

Early stage of acceleration:
Physical cause of injection and formation of quasi-monoenergetic bunch
Parameters of numerical demonstration

(Texas Petawatt laser*)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulse power</td>
<td>1.33 PW</td>
</tr>
<tr>
<td>Wavelength</td>
<td>1.057 μm</td>
</tr>
<tr>
<td>Density</td>
<td>$2.5 \times 10^{17}$ cm$^{-3}$ → $\gamma_g = \lambda_p / \lambda_0 = 63.25$ ($P/P_{cr} = 20$)</td>
</tr>
<tr>
<td>Spot size</td>
<td>27.4 μm</td>
</tr>
<tr>
<td>Pulse FWHM</td>
<td>150 fs</td>
</tr>
<tr>
<td>Plasma length</td>
<td>1.5 cm</td>
</tr>
</tbody>
</table>

Laser is strongly mismatched for self-guiding:

$$k_pr_0 \approx 2.6 << 2(a_0)^{1/2} \approx 6.2$$

---

Laser diffracts
Bubble expands
Electrons injected

Laser self-guides
Bubble steadily contracts
Electrons accelerated

\[ \times 10^{20} \]

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    xlabel={z (cm)},
    ylabel={Peak Intensity (W/cm²)},
    xmin=0, xmax=1.5,
    ymin=0, ymax=1.2,
    xtick={0,0.5,1,1.5},
    ytick={0,0.4,0.8,1.2},
    grid=both,
    legend entries={a,b,c},
    legend style={at={(0.05,0.95)},anchor=north west},
]
\addplot[red,smooth] coordinates{(0,1.2) (0.5,0.8) (1.5,1.2)} node [above] at (axis cs:0.5,0.8) {b};
\addplot[black,smooth] coordinates{(0,0.4) (0.5,0.8) (1.5,0.4)} node [above] at (axis cs:0.5,0.8) {a};
\addplot[black,smooth] coordinates{(0,0.4) (0.5,0.8) (1.5,0.4)} node [above] at (axis cs:0.5,0.8) {c};
\end{axis}
\end{tikzpicture}
\end{center}

\textbf{WAKE simulation}

**In both WAKE and CALDER-Circ runs:**

\textbf{a → b:} Pulse diffracts as in vacuum. Expanding bubble traps \(~10^{10}\) electrons (CALDER-Circ)

\textbf{b → c:} Laser self-guides. Contraction of the bubble extinguishes injection and truncates the tail of electron bunch

Injection begins and terminates at the same positions in both test-particle and full PIC simulation

\[ \rightarrow \text{beam loading has no effect on initiation and termination of injection} \]

Beam loading in CALDER-Circ reduces acceleration gradient and slows down phase space rotation
Beam loading

**Tip of the bunch:** $\Delta E_1 \approx 0$

CALDER macro-particles and WAKE test particles have the same energy

**Tail:** $\Delta E_2 > 0$

Transverse fields of the bunch (C.-c.) perturb the sheath and reduce accelerating gradient

**Electron bunch in CALDER-Circ modeling:**

$Q = 1.3 \text{ nC} \quad \sigma_{\text{r.m.s.}} = 5 \ \mu\text{m} \quad L_b = 15 \ \mu\text{m}$

Sheath electrons cross the axis

$\Rightarrow$ bubble remains closed until

$R^4_b / (8r_t^2) > \Lambda_0 \approx (\sigma_{\text{r.m.s.}}^2 / 2)(n_{b0} / n_0)$


$R^4_b / (8r_t^2 \Lambda_0) \approx 5.3$

Beam loading alone is unable to extinguish self-injection
Uncontrolled acceleration through dephasing:

Physical origin of dark current and manifestation of beam loading
Uncontrolled acceleration through dephasing: Two stages of the process

Stage I
The pulse self-guides with minimal oscillations in the waist size, which cause injection and rapid formation of the QM bunch; the pulse leading edge accumulates large red-shift.

Stage II
Self-steepening of the leading edge transforms the pulse into a single-cycle relativistic optical shock.

The bubble constantly elongates, causing continuous self-injection, contaminating electron spectrum with a polychromatic tail.

Beam loading is unable to saturate injection before dephasing.

Bucket expansion is mostly unrelated to the beam loading.

Parameters are standard for experiments with sub-100 TW laser facilities

<table>
<thead>
<tr>
<th>Wave-length, $\lambda_0$</th>
<th>Power/energy</th>
<th>TLD (FWHM in intensity)</th>
<th>Spot size, $r_0$</th>
<th>Plasma density, $n_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8 (\mu\text{m})</td>
<td>70 TW / 2.1 J</td>
<td>30 fs</td>
<td>13.6 (\mu\text{m})</td>
<td>(6.5\times10^{18}\ \text{cm}^{-3})</td>
</tr>
</tbody>
</table>
Stage I (formation of QME bunch)

Pulse self-focuses, its head remains guided, **tail flops inside the bubble**.

⇒ Bubble size varies
⇒ Electrons are injected from the sheath

**Bubble expands**: injection is continuous with broad spectrum

**Bubble contracts**: injection terminates; phase space rotation produces monochromatic energy spectrum

Beam loading enhances bubble expansion by ~30%, but is unable to prevent/delay stabilization and contraction

BL slows down phase space rotation and reduces energy gain by ~20%

Collection volume is unaltered (**electrons are injected from sheath only**)

The bubble thus remains predominantly quasi-static.
Stage II: acceleration through dephasing

Laser pulse self-compresses; bubble continuously expands; continuous injection produces polychromatic tail: \( Q_{QME} / Q_{tail} \approx 6.5 \)

Collection volume is unaltered (electrons are still injected from the sheath only)!

Near dephasing, beam loading enhances bubble expansion by merely \(~ 40\%\)

Thus, it is not the dominant cause of expansion!


The bubble thus remains predominantly quasi-static through dephasing.

Beam loading as a cause of continuous injection is ruled out!
As the relativistic optical shock builds up,

- the bubble slows down
- the dephasing length shortens
- the energy gain drops

Pulse self-compression limits electron energy gain in the blowout regime

Negative charge builds up inside the optical shock ("snow-plow" effect)

- sheath electrons – injection candidates – become exposed to higher longitudinal electric field due to stronger charge separation
- sheath electrons receive stronger backward push ⇒ their return to axis is delayed
- bubble expands: massive continuous self-injection ensues

Simulation code: WAKE
A proper choice of the laser pulse phase, compensating for the nonlinear red-shift, shall delay formation of the optical shock through electron dephasing, (i) keeping the beam clean of the low-energy background, (ii) increasing the dephasing length, boosting electron energy.

1. Bandwidth requirements: The nonlinear red-shift due to wake excitation is of order $\omega_0$. To compensate, the pulse bandwidth must approach the carrier frequency!

2. Phase profile: The red-shift is localized at the pulse leading edge. Advance higher frequencies in time – introduce the negative frequency chirp!

Additional measures:

3. Propagating the pulse in a channel

   (a) suppresses diffraction of the pulse leading edge, further delaying self-steepening, and

   (b) causes controllable periodic injection and multi-bunching in a phase space, generating comb-like electron beams

**Initial conditions:**

**Transform-limited Gaussian pulse**

![Transform-limited Gaussian pulse graph](image)

\[ |a|^2 \]

\[ (\omega - \omega_0)/\omega_0 \]

\[ \xi \equiv z - ct (\mu m) \]

**Negatively chirped Gaussian pulse**

![Negatively chirped Gaussian pulse graph](image)

\[ |W(\xi, \omega)| \]

\[ (\omega - \omega_0)/\omega_0 \]

\[ z - ct (\mu m) \]

**Wigner transform**: quasi-probability that helps estimate the photon density in phase space \((\omega, \xi)\), providing the measure for the local frequency shift

\[
W(\xi, \omega, r = 0, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(\xi + \frac{1}{2} \xi', z)a^*(\xi - \frac{1}{2} \xi', z)e^{i(\delta\omega/c)\xi'} d\xi'
\]
Local frequency shifts, pulse front etching, suppression of bubble expansion (WAKE)

When incident pulse is negatively chirped,
- optical shock does not form;
- pulse average frequency remains high;
- quasistatic bubble expansion is greatly reduced

⇒ Pulse speeds up
⇒ Electron dephasing is delayed
⇒ Energy gain of the QME bunch (test electrons) nearly doubles
⇒ Flux in the tail drops nearly twice
**CALDER-Circ simulation:**

**Bubble size vs propagation distance**
- Chirp & taper
- No chirp & no taper

**Lorentz factor (tail of the bubble)**

**“Collection phase space”** ($p_z \text{fin} \text{ vs } z_{\text{in}}$)

**Collection volume**

Effect of the chirp

**Polychromatic background:**
- Charge drops nearly by half
- Average flux (charge per MeV) drops by a factor of 3.5

**QME bunch:**
- Energy doubles, reaching 1 GeV
- Relative energy spread drops by a factor 2.5
- Brightness increases by 60%
Dark current control in the strongly nonlinear interaction regime (200 TW, 55 fs pulse, \(n_{e0}=5.7\times10^{18}\text{cm}^{-3}\))

\[
P/P_{cr} \approx 40, \; \omega_{pe}\tau_{FWHM} \approx 7.4
\]

Dephasing point for electrons in the chirped-pulse simulation

\(z \approx 4.05\) mm

\(z \approx 5.75\) mm

Electron beams far beyond dephasing:

<table>
<thead>
<tr>
<th>(z = 5.75) mm</th>
<th>(Q_{qm}) nC</th>
<th>(E) MeV</th>
<th>(\Delta E) MeV</th>
<th>(\Delta E/E)</th>
<th>(\varepsilon_{N,x}) mm mrad</th>
<th>(\varepsilon_{N,y}) mm mrad</th>
<th>(&lt;B&gt;_{qm})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chirp</td>
<td>3.4</td>
<td>640</td>
<td>80</td>
<td>0.125</td>
<td>38.4</td>
<td>38.4</td>
<td>1.75 x 10^{11}</td>
</tr>
<tr>
<td>No chirp</td>
<td>3.1</td>
<td>750</td>
<td>250</td>
<td>0.3</td>
<td>60</td>
<td>60</td>
<td>0.25 x 10^{11}</td>
</tr>
</tbody>
</table>

Negatively chirped driver makes even continuous injection much quieter!
**Summary**

*Quasistatic response of bulk plasma electrons* to the steadily varying ponderomotive force of the evolving laser pulse is the cause of electron self-injection and the main physical phenomenon behind the electron beam phase space evolution in the blowout regime of laser wakefield acceleration.

Using reduced computational models allowing to “turn off” the fields and currents of injected electrons (viz. quasistatic WAKE code with fully dynamic test particles), in combination with full 3D PIC codes, makes it possible to find that

a) The bubble evolution (and, hence, dynamics of self-injection) remains almost unaffected by the beam loading through electron dephasing.

b) Early termination of self-injection is thus not associated with the beam loading.

c) The beam loading slows down phase space rotation and early formation of the quasi-monoenergetic beam.

d) Continuous expansion and massive self-injection, contaminating the electron spectrum with a high-charge, polychromatic tail, is a consequence of laser pulse self-compression, rather than of beam loading.

e) Negative chirp of the pulse (giving the pulse bandwidth equivalent to < 1.5 optical-cycle transform-limited duration) avoids formation of the optical shock and drastically reduces continuous injection.
Supplements
Standard results of acceleration until dephasing at $n_{e0} \geq 6 \times 10^{18}$ cm$^{-3}$ with 50-60 fs, 60-200TW laser pulses: poor quality electron spectra with large polychromatic background

Common two-stage scenario of electron injection:

All simulations presented in this slide were made with fully explicit, 3D PIC codes
Monoenergetic electron beam optimization in the bubble regime

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(Received 16 November 2004; accepted 12 January 2005; published online 7 April 2005)

Within the last decade, laser-plasma-based accelerators have been able to deliver electron beams with Maxwellian energy distributions characterized by effective temperatures in the range of 1–20 MeV. Changing the interaction parameters, the electron beam quality was improved. Especially, matching the interaction length to the dephasing length was crucial to produce an extremely high quality electron beam with a quasimonoenergetic distribution at 170 MeV. The optimization of these distributions is presented, as well as comparisons with three-dimensional particle-in-cell (PIC) simulations. © 2005 American Institute of Physics. [DOI: 10.1063/1.1869498]

III. NUMERICAL SIMULATIONS

To get more insight into the mechanism of electron acceleration in plasma leading to the monoenergetic beams, we have run several 3D particle-in-cell (PIC) simulations using the code Virtual Laser Plasma Laboratory. In the simulations, the laser pulse duration was 30 fs and the laser pulse energy 1 J. We started with the wide focal spot of 21 μm FWHM. The simulations were done for three plasma densities: $n_1 = 3 \times 10^{18}$ cm$^{-3}$, $n_2 = 6 \times 10^{18}$ cm$^{-3}$, $n_3 = 1.2 \times 10^{19}$ cm$^{-3}$.

The optimal case corresponded to the plasma density $n_2 = 6 \times 10^{18}$ cm$^{-3}$. The simulation reveals that the laser pulse self-focuses as it propagates in the plateau region of the gas jet. As the effective radius of the laser pulse decreases, the laser intensity increases and finally becomes sufficient to generate the bubble. The laser ponderomotive force expels the plasma electrons radially and leaves a cavitated region behind the pulse. Figure 5 shows the bubble structure after

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**Misinterpretation due to exclusive use of a single model: PIC code**
Simulation of monoenergetic electron generation via laser wakefield accelerators for 5–25 TW lasers

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(Received 16 November 2005; accepted 31 March 2006; published online 26 May 2006)

In 2004, using a 3D particle-in-cell (PIC) model [F. S. Tsung et al., Phys. Rev. Lett. 93, 185004 (2004)], it was predicted that a 16.5 TW, 50 fs laser propagating through nearly 0.5 cm of $3 \times 10^{11}$ cm$^{-2}$ preformed plasma channel would generate a monoenergetic bunch of electrons with a central energy of 240 MeV after 0.5 cm of propagation. In addition, electrons out to 840 MeV were seen if the laser propagated through 0.8 cm of the same plasma. The simulations showed that self-injection occurs after the laser intensity increases due to a combination of photon deceleration, group velocity dispersion, and self-focusing. The monoenergetic beam is produced because the injection process is clamped by beam loading and the rotation in phase space that results as the beam dephases. Nearly simultaneously [S. P. D. Mangles et al., Nature 431, 538 (2004); C. G. R. Geddes et al., ibid. 434, 541 (2004)]; J. Fauve et al., ibid. 434, 538 (2004)]; three experimental groups from around the world reported the generation of near nano-Coulomb of low emittance, monoenergetic electron beams using similar laser powers and pulse lengths as those reported in our simulations. Each of these experiments is modeled using the same 3D PIC code OSIRIS. The simulations indicate that although these experiments use a range of plasma parameters, density profiles, laser powers, and spot sizes, there are some commonalities to the mechanism for the generation of monoenergetic beams. Converges on how the energy and beam quality can be improved in the future. © 2006 American Institute of Physics. [DOI: 10.1063/1.2198535]

Accurate physical picture leading to the correct theory of beam loading in the blowout regime

Misinterpretation
On the stability of laser wakefield electron accelerators in the monoenergetic regime

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The effects of plasma density and laser energy on the stability of laser produced monoenergetic electron beams are investigated. Fluctuations in the principal beam parameters, namely, electron energy, energy-spread, charge, and pointing, are demonstrated to be minimized at low densities. This improvement in stability is attributed to the reduced time for pulse evolution required before self-injection occurs; i.e., that the pulse is closest to the matched conditions for these densities. This is also observed that electrons are only consistently produced above a density-dependent energy threshold. These observations are consistent with there being a threshold intensity ($I_0 > 3$) required for the occurrence of self-injection after accounting for pulse compression. © 2007 American Institute of Physics. [DOI: 10.1063/1.2436481]

The standard picture of self-injection in multidimensional nonlinear plasma waves is as follows: after the ponderomotive force of the laser pulse expels electrons at the front of the plasma wave bubble, electrons return laterally at the back of the bubble in response to the charge imbalance. In the wave frame, electrons that have low longitudinal velocity are injected at the rear of the bubble. Those electrons that are injected first are accelerated ahead of those injected last until they reach the front of the plasma wave and are deflected. At this point, the tail of the bunch approaches the same energy as the head of the bunch. In this simple model, one would expect the energy-spread to be minimized when the electron beam energy is close to the dephasing energy. For this simple model to remain valid requires that the plasma wave electric field strength is constant over the duration of injection. The plasma wave amplitude can be affected by beam loading and laser pulse evolution. Simulations and experimental evidence show that the electron bunch is less than a plasma wavelength in length ($<25$ fs), and simulations show that significant pulse evolution occurs over many $1/\omega_p$. It is therefore unlikely that the change in the wakefield amplitude due to pulse evolution is the cause for the observed increase in the beam energy spread as the beam energy approaches $W_{\text{max}}$.

Beam loading, i.e., the flattening of the laser wake due to the space charge of the electron bunch, will be more significant at higher densities, since the accelerated bunch has more charge, but beam loading is also unlikely to lead to increased energy spread. Once the plasma wakefield reaches the threshold for self-injection, electrons are continuously injected into the wave. The major effect of beam loading is to lower the electric field at the back of the bubble. If there is any significant lowering of the electric field then trapping can no longer occur. The cessation of trapping leads to narrow-energy-spread, short duration electron bunches. The flattening of the wake electric field due to beam loading results in the maximum energy of electrons injected later being lower than the maximum energy of those injected earlier. This may result in some increase in the energy spread but is unlikely to lead to the very large energy spreads observed when the beam energy approaches $W_{\text{max}}$. This suggests an additional mechanism is causing longitudinal energy spread in the accelerated electron bunch.

Statement impossible to support or refute on the basis of fully kinetic simulations.
Statements impossible to support or refute on the basis of fully kinetic simulations.
Progress of the bubble and electron beam through the Stage II (CALDER-Circ simulation)

No chirp

Chirp

No chirp

Chirp

\[ E (\text{GeV}) \]

\[ p_z (10^3 \text{m}_e) \]

\[ z (\text{mm}) \]

\[ \frac{dN}{dE} (10^6 \text{MeV}^{-1}) \]

\[ 0 \quad 25 \quad 50 \quad 75 \]

\[ 0.05 \quad 0.2 \quad 0.4 \]

\[ 0.05 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \]

\[ 0.05 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \]
Red – CALDER-Circ; black – WAKE

Black – CALDER-Circ; color – VORPAL with perfect dispersion
[B. M. Cowan et al., J. Plasma Phys. (2012); ]